JOURNAL OF ALGEBRA 4, 365-372 (1966)

Grushko's Theorem

P. J. HIGGINS

King's College, London, England

Communicated by A. Frohlich

Received July 21, 1965

1. INTRODUCTION

A new proof of Grushko's theorem is given based on the theory of groupoids developed by P. J. Higgins [*Proceedings of the Cambridge Philosophical Society* **60** (1964), 7–20]. A topological proof of this theorem has been given by J. R. Stallings in *Math. Z.* **90**(1965), 1–8, and, like the topological proofs of the subgroup theorems of Schreier and Kurosh, it makes implicit use of the fundamental groupoids of various spaces. However, it is entirely irrelevant that these groupoids arise from topological spaces and, as often happens when irrelevancies are discarded, much simpler lines of argument emerge in the context of abstract groupoids. The argument given here is the result of such simplification and has little in common with Stallings' proof. Its natural endproduct is the following theorem on groups.

THEOREM 1. *Let G and B be groups with free decompositions $G = *_{\lambda\in\Lambda} G^\lambda$, $B = *_{\lambda\in\Lambda} B^\lambda$. Let ψ: $G \to B$ be an epimorphism mapping G^λ to B^λ, for all $\lambda \in \Lambda$. If H is any subgroup of G such that $H\psi = B$, then H has a decomposition $H = *_{\lambda\in\Lambda} H^\lambda$ such that $H^\lambda\psi = B^\lambda$.*

From this it is easy to deduce the generalized form of Grushko's theorem given by D. H. Wagner ([*3*], Theorem 4.4):

COROLLARY. *Let ϕ be an epimorphism from a free group F to a free product $B = *_{\lambda\in\Lambda} B^\lambda$ of groups. Then F has a decomposition $F = *_{\lambda\in\Lambda} F$ such that $F^\lambda\phi = B^\lambda$.*

Proof. It will be sufficient, by the theorem, to construct a group $G = * G^\lambda$, an epimorphism $\psi : G \to B$ mapping G^λ to B^λ for all $\lambda \in \Lambda$, and a monomorphism $\sigma : F \to G$ such that $\sigma\psi = \phi$. For then the subgroup $H = F\sigma$ has image $H\psi = F\phi = B$, whence $H = * H^\lambda$ with $H^\lambda\psi = B^\lambda$, and the conclusion follows with $F^\lambda = H^\lambda\sigma^{-1}$. We give one such construction, perhaps the most natural.

Let X be a set of free generators of F, and for each $x \in X$, let $x\phi = b_{x1}b_{x2} \cdots b_{xr}$, where $r = r(x) \geqslant 1$ and each b_{xi} lies in some B^λ. This product need not be the canonical form of x (indeed it cannot be if $x\phi = 1$), but we take a fixed such product for each $x \in X$. Now let Y be a set whose members are distinct symbols $y_{xi}(x \in X,\ i = 1, 2, \ldots, r(x))$ and let G be the free group on Y. We can partition Y into disjoint sets Y^λ so that $y_{xi} \in Y^\lambda$ implies $b_{xi} \in B^\lambda$. Then $G = * G^\lambda$, where G^λ is generated by Y^λ, and the homomorphism $\psi : G \rightarrow B$ defined by $y_{xi}\psi = b_{xi}$ maps G^λ to B^λ. Finally, the homomorphism $\sigma : F \rightarrow G$ defined by $x\sigma = y_{x1}y_{x2} \cdots y_{xr}$ is a monomorphism (since the y_{xi} are distinct free generators of G) and clearly $\sigma\psi = \phi$.

Theorem 1 will be deduced from a theorem on groupoids (Theorem 2, below), and we refer to [*1*] for the basic definitions. However, this opportunity is taken to refine some of the concepts and results of [*1*]; this has the advantage of making the paper more self-contained.

2. Projections

In order to facilitate the discussion of groupoids with different vertex sets, we depart from the terminology of [*1*] in two respects. First, we shall think of a groupoid as the union $A = \bigcup_{i,j\in I} A_{ij}$ of disjoint sets A_{ij}, rather than as the family $\{A_{ij}\}$. A groupoid is thus the set of morphisms of an abstract category (in which all morphisms are invertible); and a homomorphism of groupoids (functor) is a single map rather than a family of maps as in [*1*]. The set A is understood to carry the structure of a directed graph (vertex = object, edge = morphism) and a partial multiplication whose domain is determined by the graph.

Second, the definition of normal subgroupoid in [*1*] is too restricted, and we replace it by the following: the subgroupoid N of A is ***normal*** if (i) it contains all identity elements of A, and (ii) $a^{-1}xa \in N_{jj}$ whenever $x \in N_{ii}$ and $a \in A_{ij}$. The construction of quotient groupoids can now be generalized. Let N be a normal subgroupoid of A, and write $a \equiv b \pmod N$ if there exist $x,\ y \in N$ such that $a = xby$. This is an equivalence relation on A, and the equivalence classes $\bar{a}$ form a directed graph $\bar{A}$ whose vertices $\bar{\imath}$ are in one-one correspondence with the connected components of N. If $\bar{a}$, $\bar{b}$ are edges in $\bar{A}$ joining $\bar{\imath}$ to $\bar{\jmath}$ and $\bar{\jmath}$ to $\bar{k}$, respectively, then there exist $a' \equiv a$ and $b' \equiv b \pmod N$ with $a' \in A_{ij}$, $b' \in A_{jk}$, and we may define $\bar{a}\bar{b} = \overline{a'b'}$. The normality of N ensures that the definition is unambiguous, and with this multiplication $\bar{A}$ becomes a groupoid, denoted by A/N. The ***quotient map*** $a \rightsquigarrow \bar{a}$ is a homomorphism π: $A \rightarrow A/N$ with kernel N. (The ***kernel*** of a homomorphism θ: $A \to B$ is the subgroupoid $\operatorname{Ker}\theta$ of A consisting of all elements which map to identity elements of B; it is always a normal subgroupoid.)

We recall that if θ: $A \to B$ is a homomorphism with kernel N, then $A\theta$ is not in general a subgroupoid of B and its structure is not determined by N. However, if π: $A \to A/N$ is the quotient map, then θ induces a unique homomorphism θ^*: $A/N \to B$ such that $\theta = \pi\theta^*$. ($\operatorname{Ker}\theta^*$ is *trivial*, i.e., it consists of identity elements only). If θ^* is an isomorphism $A/N \cong B$, we shall say that θ is a *projection*.

PROPOSITION 1. *If $\theta : A \to B$ is a projection with kernel N, and ϕ: $A \to C$ is any homomorphism whose kernel contains N, then there is a unique homomorphism ϕ^*: $B \to C$ such that $\phi = \theta\phi^*$.*

Proof. If $a\theta = a'\theta$, then $a \equiv a'$ (mod N) since θ is a projection. Thus $a = xa'y$ with $x, y \in N$, and $a\phi = a'\phi$ since $x\phi$ and $y\phi$ are identity elements of C. It follows that there is a unique mapping ϕ^*: $B \to C$ with $\phi = \theta\phi^*$, and it is clear that ϕ^* is a homomorphism.

PROPOSITION 2. *If θ: $A \to B$ and ϕ: $B \to C$ are projections, then so is $\theta\phi$: $A \to C$.*

Proof. Let $\psi = \theta\phi$, $N = \operatorname{Ker}\psi$, and let π be the quotient map $A \to A/N$. Then ψ induces ψ^*: $A/N \to C$, and we have to show that ψ^* is an isomorphism. Since θ is a projection and $N = \operatorname{Ker}\pi \supset \operatorname{Ker}\theta$, there is a homomorphism $\pi_1 : B \to A/N$ such that $\pi = \theta\pi_1$. Now $\operatorname{Ker}\pi_1 \supset N\theta = \operatorname{Ker}\phi$, and ϕ is a projection, so there is a homomorphism $\pi^* : C \to A/N$ with $\pi_1 = \phi\pi^*$. Since π and ψ are surjective, and $\pi\psi^* = \psi$, $\psi\pi^* = \pi$, it follows that π^* is inverse to ψ^*.

3. Homomorphisms of Free Products

The definition of free product in [*1*] is appropriate for the discussion of groupoids on a fixed set of vertices. We adopt here a slightly different definition which is easily seen to be equivalent to it (cf. [*1*], Theorem 6). A groupoid A is the *free product* of subgroupoids A^λ ($\lambda \in \Lambda$) if, whenever $\alpha^\lambda : A^\lambda \to C$ are homomorphisms *which agree on common identity elements of the* A^λ, there is a unique homomorphism $\alpha : A \to C$ whose restriction to A^λ is α^λ (for all $\lambda \in \Lambda$). We write $A = *_{\lambda\in\Lambda} A^\lambda$ or simply $A = * A^\lambda$.

Suppose that $A = * A^\lambda$ and $\theta : A \to B$ is a homomorphism. We say that *θ preserves the free decomposition $A = * A^\lambda$* if the groupoid generated by $A\theta$ is the free product of the groupoids generated by the $A^\lambda\theta$. For groups, this is equivalent to saying that Ker θ is generated, as normal subgroup of A, by its intersections with the A^λ. For groupoids, this condition is neither necessary nor sufficient. We prove partial results in each direction.

PROPOSITION 3. *Let* $A = * A^\lambda$, *and let* $\theta : A \to B$ *be a projection with kernel* N. *If* N *is generated, as normal subgroupoid of* A, *by the subgroupoids* $N^\lambda = N \cap A^\lambda$, *then* θ *preserves the decomposition* $A = * A^\lambda$.

Proof. Write $B^\lambda = Gpd\{A^\lambda\theta^\lambda\}$, where θ^λ is the restriction of θ to A^λ. We have to show that $B = * B^\lambda$. (Note that $\theta^\lambda : A^\lambda \to B^\lambda$ is not, in general, a projection.) Let $\beta^\lambda : B^\lambda \to C$ be homomorphisms agreeing on common identities of the B^λ. Then the homomorphisms $\alpha^\lambda = \theta^\lambda\beta^\lambda : A^\lambda \to C$ agree on common identities of the A^λ; hence there is a unique homomorphism $\alpha : A \to C$ extending the α^λ. Since $\operatorname{Ker}\alpha \supset \operatorname{Ker}\alpha^\lambda \supset N^\lambda$ for all λ, we have $\operatorname{Ker}\alpha \supset N$, and by Proposition 1, there is a unique $\beta : B \to C$ such that $\alpha = \theta\beta$. Since B^λ is generated by $A^\lambda\theta^\lambda$, the restriction of β to B^λ must coincide with β^λ, and β is clearly unique with this property.

PROPOSITION 4. *Let* $A = * A^\lambda$, *and suppose that, for each* $\sigma \in \Sigma$, $\theta_\sigma : A \to B_\sigma$ *is a projection with kernel* N_σ. *Let* $\theta : A \to B$ *be a projection whose kernel* N *is the normal subgroupoid of* A *generated by all the* N_σ. *If each* θ_σ *preserves the decomposition* $A = * A^\lambda$, *then so does* θ.

Proof. Write $B^\lambda = Gpd\{A^\lambda\theta^\lambda\}$, where θ^λ is the restriction of θ to A^λ, and let $\beta^\lambda : B^\lambda \to C$ be homomorphisms agreeing on common identities of the B^λ. As in the proof of Proposition 3, there is a unique homomorphism $\alpha: A \to C$ whose restriction to A^λ is $\alpha^\lambda = \theta^\lambda\beta^\lambda$, and it is enough to show that $\operatorname{Ker}\alpha \supset N$. It is therefore enough to show that $\operatorname{Ker}\alpha \supset N_\sigma$ for an arbitrary $\sigma \in \Sigma$.

Take a fixed $\sigma \in \Sigma$, and write $B_\sigma{}^\lambda = Gpd\{A^\lambda\theta_\sigma{}^\lambda\}$, where $\theta_\sigma{}^\lambda$ is the restriction of θ_σ to A^λ. By Proposition 1, $\theta : A \to B$ induces a homomorphism $\phi_\sigma : B_\sigma \to B$ with $\theta_\sigma\phi_\sigma = \theta$, and ϕ_σ induces homomorphisms $\phi_\sigma{}^\lambda : B_\sigma{}^\lambda \to B^\lambda$ by restriction. Since, by hypothesis, $B_\sigma = *_{\lambda\in\Lambda} B_\sigma{}^\lambda$, the homomorphisms $\phi_\sigma{}^\lambda\beta^\lambda : B_\sigma{}^\lambda \to C$ induce a unique homomorphism $\alpha_\sigma : B_\sigma \to C$. Consider now the homomorphism $\theta_\sigma\alpha_\sigma : A \to C$. Its restriction to A^λ is $\theta_\sigma{}^\lambda\phi_\sigma{}^\lambda\beta^\lambda = \theta^\lambda\beta^\lambda = \alpha^\lambda$; so, by the uniqueness of α, we have $\theta_\sigma\alpha_\sigma = \alpha$. It follows that $\operatorname{Ker}\alpha \supset \operatorname{Ker}\theta_\sigma = N_\sigma$, as required.

DEFINITION. *A groupoid is* totally disconnected *if it is the disjoint union of groups (one at each vertex).*

PROPOSITION 5. *Let* $A = * A^\lambda, B = * B^\lambda$, *and suppose that the homo-*

morphism $\theta : A \rightarrow B$ *maps* A^λ *into* B^λ *for each* λ. *Let* Ker $\theta = K$. *If* $A^\lambda \cap K$ *is totally disconnected for each* λ, *then* K *is totally disconnected.*

Proof. Suppose that K is not totally disconnected, and choose vertices $i \neq j$ of A such that $K_{ij} \neq \varnothing$. Let x be an element of K_{ij} of minimal length with respect to the decomposition $A = * A^\lambda$ (we refer to [*1*], p. 15, for the solution of the word problem for free products of groupoids). Then $x = x_1 x_2 \cdots x_n$ with $x_\nu \in A^{\lambda_\nu}$, $\lambda_\nu \neq \lambda_{\nu+1}$ and n minimal. Since $i \neq j$ and each $A^\lambda \cap K$ is totally disconnected, we have $n \geqslant 2$. Write $\bar{x}_\nu = x_\nu \theta$. Then $\bar{x}_\nu \in B^{\lambda_\nu}$, and $\bar{x}_1 \bar{x}_2 \cdots \bar{x}_n$ is an identity element of B. Since $B = * B^\lambda$ we deduce that some $\bar{x}_\nu$ is an identity element, whence x_ν lies in $A^{\lambda_\nu} \cap K$, which is totally disconnected. Thus x_ν lies in some vertex group of A and if we omit it, the product $x' = x_1 x_2 \cdots \hat{x}_\nu \cdots x_n$ is still defined in A. Clearly, x' lies in K_{ij} and is shorter than x. This contradiction establishes the result.

4. The Groupoid Theorem

We shall call a groupoid $G = \bigcup_{i,j \in I} G_{ij}$ *elementary* if each G_{ij} has at most one element. Equivalently, G is elementary if all its vertex groups are trivial. Theorem 5 of [*1*] asserts that any groupoid is the free product of an elementary groupoid and a totally disconnected groupoid, and this separates the graph-theoretical and the group-theoretical aspects of the groupoid structure. Appropriate choice of such a decomposition in a given free product of groupoids leads to the Kurosh subgroup theorem (see [*1*], p. 17). We adopt a similar procedure here, but the crucial result is a factorization of a given homomorphism.

Theorem 2. *Let* $A = * A^\lambda$, $B = * B^\lambda$, *and suppose that the homomorphism* θ: $A \rightarrow B$ *maps* A^λ *into* B^λ *for each* λ. *Then* θ *has a factorization* $\theta = \rho\theta^*$ ($\rho : A \rightarrow C$, $\theta^* : C \rightarrow B$) *with the following properties*:

(i) ρ *is a projection*;

(ii) ρ *preserves the decomposition* $A = * A^\lambda$;

(iii) Ker ρ *is elementary*;

(iv) Ker θ^* *is totally disconnected.*

Proof. An elementary subgroupoid N of A is normal if and only if it contains all the identity elements of A. Let $\mathscr{S}$ be the set of all elementary normal subgroupoids N of A such that $N \subset$ Ker θ and such that the projection $\pi : A \rightarrow A/N$ preserves the decomposition $A = * A^\lambda$. $\mathscr{S}$ is ordered by inclusion and is non-empty. The union of a chain in $\mathscr{S}$ is again an elementary normal subgroupoid of A contained in Ker θ, and Proposition 4 ensures that it lies in $\mathscr{S}$. It follows, by Zorn's lemma, that $\mathscr{S}$ has a maximal member M. Let ρ be the quotient map $A \to A/M = C$. Then ρ has properties (i), (ii), and (iii), and since $M \subset \operatorname{Ker} \theta$, there is a unique homomorphism θ^*: $C \to B$ such that $\theta = \rho\theta^*$. It remains to show that $\operatorname{Ker} \theta^*$ is totally disconnected. Now $C = * C^\lambda$, where $C^\lambda = Gpd\{A^\lambda \rho\}$, and θ^* maps C^λ into B^λ for each λ. Suppose that $K = \operatorname{Ker} \theta^*$ is not totally disconnected. Then, by Proposition 5, some $K \cap C^\lambda$ is not totally disconnected, so there is an element x in some $K \cap C^\lambda$ which joins distinct vertices of C. The elements x, x^{-1} and the identity elements of C form an elementary normal subgroupoid X of C which is certainly generated by its intersections with the subgroupoids C^λ. Hence, by Proposition 3, the projection $\xi : C \to C/X$ preserves the decomposition $C = * C^\lambda$. Now $\rho' = \rho\xi : A \to C/X$ is a projection (Proposition 2) and it preserves the decomposition $A = * A^\lambda$. Since ρ and ξ have elementary kernels, so does ρ' (a homomorphism has elementary kernel if and only if it maps each vertex group injectively). Finally, since $\operatorname{Ker} \xi = X \subset \operatorname{Ker} \theta^*$, we have $\operatorname{Ker} \rho' \subset \operatorname{Ker} \theta$. Thus $M' = \operatorname{Ker} \rho'$ is in $\mathscr{S}$. But clearly M' properly contains M since M' contains a counter-image of the element x of C. This contradiction proves the theorem.

COROLLARY. *If in Theorem 2, A is connected, B is a group and $V\theta = B$ for some vertex group V of A, then C is a group and ρ induces an isomorphism $V \to C$. Hence V has a decomposition $V = * V^\lambda$ with $V^\lambda\theta = B^\lambda$ for each λ.*

Proof. $\rho : A \to C$ is surjective, so C is connected. Also $V\rho$ is contained in a vertex group C_{11} of C, so $C_{11}\theta^* = B$. These two facts imply that $K = \operatorname{Ker} \theta^*$ is connected. (Let i be any vertex of C, and choose $x \in C_{i1}$. There is an element $c \in C_{11}$ with $c\theta^* = x\theta^*$, so $xc^{-1} \in K_{i1}$). But, according to the theorem, K is totally disconnected, and it follows that C has only one vertex, i.e., C is a group. Now $\rho : A \to C$ is a projection onto a group, so $M = \operatorname{Ker} \rho$ is connected. M is also elementary, and it is easy to verify that under these circumstances the vertex group V contains exactly one element of each class (mod M). Hence ρ induces an isomorphism $V \to C$, and the decomposition $C = * C^\lambda$ gives the required decomposition $V = * V^\lambda$ with $V^\lambda\theta \subset B^\lambda$. Since $V\theta = B$, we must have $V^\lambda\theta = B^\lambda$ for each λ.

5. Proof of Theorem 1

We are given an epimorphism $\psi : G \to B$, where $G = * G^\lambda$, $B = * B^\lambda$ are groups and $G^\lambda\psi = B^\lambda$ for all λ. For any subgroup H of G we can form the groupoid $A = \operatorname{Tr}(G : H)$ whose objects (vertices) are the right cosets of H in G and whose morphisms (edges) are the right translations $x \rightsquigarrow xg$ of these cosets by elements g of G. Then the natural homomorphism γ :

$A \to G$ induces a free decomposition $A = * A^\lambda$ of A, where $A^\lambda = G^\lambda \gamma^{-1}$. (See [*1*], Theorem 10, and the generalization to covering homomorphisms in § 6 below.) Putting $\theta = \gamma\psi : A \to B$, we are in the situation of Theorem 2, with A connected and B a group. Now H can be embedded in A as a vertex group V by the map $\mu : H \to A$ which sends each element h to the right translation induced by h on H itself. Then $\mu\gamma : H \to G$ is the inclusion map, and $V\theta = H\mu\theta = H\mu\gamma\psi = H\psi = B$ (by hypothesis). The corollary to Theorem 2 now gives a decomposition $V = * V^\lambda$ with $V^\lambda\theta = B^\lambda$, and this induces, via μ, a decomposition $H = * H^\lambda$ with $H^\lambda\psi = B^\lambda$.

6. Remarks on Coverings and Retractions

(i) In the proof of Theorem 1 we used the fact (proved in [*1*]) that a free decomposition of a group G lifts to a free decomposition of the translation groupoid $\mathrm{Tr}(G : H)$. It is worth noting that this is a special case of a very natural theorem on covering groupoids.

For any groupoid A we denote by $A_{i\cdot}$ the set $\bigcup_j A_{ij}$ of edges starting from the vertex i. We then say that a homomorphism $\gamma : A \to B$ is a *covering homomorphism* if (i) each vertex of B is the image under γ of some vertex of A, and (ii) for each i, the map induced by γ from $A_{i\cdot}$ to $B_{i\gamma\cdot}$ is bijective. Covering maps of topological spaces induce covering homomorphisms of their fundamental groupoids. It is clear that if $b_1 b_2 \cdots b_n$ is a path in B starting from the vertex k (i.e., $b_1, b_2, \ldots, b_n$ are elements of B such that the product $b_1 b_2 \cdots b_n$ is defined, and $b_1 \in B_{k\cdot}$) then, for each vertex i of A with $i\gamma = k$, there is a *unique* path $a_1 a_2 \cdots a_n$ in A, starting from i, such that $a_r\gamma = b_r$ $(r = 1, 2, \ldots, n)$. From this one easily deduces the following facts:

(a) The cardinal of $b\gamma^{-1}$ is constant on the connected components of B.

(b) If B is freely generated by X, then A is freely generated by $X\gamma^{-1}$.

(c) If $B = * B^\lambda$, then $A = * A^\lambda$, where $A^\lambda = B^\lambda\gamma^{-1}$.

The natural map from $\mathrm{Tr}(G : H)$ to G is a covering homomorphism (an n-fold covering if H has index n in G) and this gives the required result. One can also lift any presentation of B to a presentation of A and obtain a generalization of Theorem 10 of [*1*].

(ii) Since homomorphisms of groupoids can be thought of as functors, one has the concept of natural equivalence, $f \simeq g$, for homomorphisms f, $g : A \to B$. This corresponds to homotopy of continuous maps; indeed, homotopic maps between topological spaces induce naturally equivalent homomorphisms between their fundamental groupoids. It is natural, therefore, to say that a groupoid homomorphism $\rho : A \to B$ is a *deformation retraction* if there is a homomorphism $\mu : B \rightarrow A$ such that $\mu\rho = 1_B$ and $\rho\mu \simeq 1_A$. (We do not insist that B be a subgroupoid of A, but μ is, of course, an injection.) It is not difficult to show that this condition on ρ is equivalent to the statement that ρ is a projection with elementary kernel. Thus the crucial map ρ in Theorem 2 is of this type. So also are the "transfers to a vertex" defined in [*1*], p. 14. The proof of Theorem 1 and the proofs of the Schreier and Kurosh subgroup theorems in [*1*] therefore have a common strategy which may be described as follows. We are given a group G and a subgroup H. Information about G is lifted to a covering groupoid A which effectively has H as vertex group. The required information about H is then obtained by means of a suitably chosen retraction from A to H.

References

1. Higgins, P. J., Presentations of groupoids, with applications to groups. *Proc. Cambridge Phil. Soc.* **60** (1964), 7-20.

2. Stallings, J. R., A topological proof of Grushko's theorem on free products. *Math. Z.* **90** (1965), 1-8.

3. Wagner, D. H., On free products of groups. *Trans. Am. Math. Soc.* **84** (1957), 352-378.